Enumeration of Graded $(3+1)$-Avoiding Posets

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Abstract

The notion of $(3+1)$-avoidance has shown up in many places in enumerative combinatorics. The natural goal of enumeration of all $(3+1)$-avoiding posets remains open. In this paper, we enumerate *graded* $(3+1)$-avoiding posets via generating functions and an application of the transfer-matrix method.

1 Introduction

The notion of $(3+1)$-avoiding posets pops up in different different areas of combinatorics, such as in the Stanley-Stembridge conjecture about the e-positivity of certain chromatic polynomials [12] and the characterization of interval semiorders [4]. Graph-theoretically, $(3+1)$-avoiding posets are exactly those posets whose comparability graphs are complements of claw-free graphs; as a result, they also are connected to a generalization of the "birthday problem" [3].

Despite these connections, the enumeration of $(3+1)$-avoiding posets has remained elusive. This is particularly bothersome because the enumeration of posets that are both $(2+2)$- and $(3+1)$-avoiding, the interval semiorders, is well-understood: the number of unlabeled n-element interval semiorders is exactly the Catalan number C_n [4]. Happily, there has been some progress: Skandera [9] has given a characterization of all $(3+1)$-avoiding posets involving the square of the antiadjacency matrix and Atkinson, Sagan and Vatter [1] have recently characterized and enumerated $(3+1)$-avoiding permutations (i.e., permutations whose associated posets are $(3+1)$-avoiding).

When we mathematicians fail to solve a problem, we usually get by by solving an easier one. In this paper, we enumerate *graded* $(3+1)$-avoiding posets (for both common meanings of the word graded) via structural theorems and generating function magic. The property of gradedness is very natural and captures a lot of the complexity of the general case while making the problem much more tractable.[1] In the rest of this introduction, we summarize our strategy and results.

[1] A substantially easier problem is to enumerate $(3+1)$- and $(2+2)$-avoiding graded posets. The solution may be found in the second-named author's Ph. D. thesis (in preparation); labeled $(3+1)$- and $(2+2)$-avoiding strongly graded posets are counted by the generating function $1+\frac{e^x(e^x-1)(e^x-2)}{e^{2x}-e^x-1}$.

In Section 2, we offer some definitions and notation that we will use throughout the paper. Then in Section 3, we give a useful local condition that is equivalent to $(3+1)$-avoidance for graded posets.

The main ideas of the paper are in Section 4, where we introduce several operations that allow us to decompose strongly graded $(3+1)$-avoiding posets into simpler objects. First, in Section 4.1 we reduce our problem of obtaining the generating function for all graded $(3+1)$-avoiding posets to studying certain posets we will call *trimmed* which are slightly simpler but which capture most of the information of the original posets. Then, in Section 4.2, we introduce the concept of *layering*, and we show that trimmed $(3+1)$-avoiding posets arise from layering together the *L-indecomposable* $(3+1)$-avoiding posets. Finally, in Section 4.3 we introduce two more operations, *gluing* and *sticking*. We show that L-indecomposable $(3+1)$-avoiding posets arise from gluing and sticking together basic units called *quarks*, which we enumerate in Section 5.

This line of argument culminates in Section 6, in which we backtrack and use the results of the preceding sections and the transfer-matrix method to enumerate all strongly graded $(3+1)$-avoiding posets. We end with some extensions of these techniques. In Section 7 we make a minor tweak to the generating functional arguments to enumerate strongly graded $(3+1)$-avoiding posets by height. We use this modified enumeration in Section 8 to enumerate $(3+1)$-avoiding *weakly* graded posets. Finally, in Section 9, we use the generating functions computed in Sections 6 and 8 to establish the asymptotic rate of growth of the number of graded $(3+1)$-avoiding posets.

2 Preliminaries

A *partially ordered set*, or *poset* for short, is a set with an irreflexive and transitive relation $>$. We say two elements a, b of a poset are *comparable* if $a > b$ or $b > a$. In this paper, we concern ourselves only with posets of finite cardinality. We say that an element w *covers* an element v, denoted $v \lessdot w$, if $v < w$ and there is no z such that $v < z < w$. Observe that the order relations of a finite poset follow by transitivity from the cover relations; this allows us to graphically represent posets by showing only the cover relations. The resulting graph is called the *Hasse diagram* of the poset.

A poset in which every pair of elements is comparable is called a *chain*, and a poset in which every pair of elements is incomparable is called an *antichain*.

We say that four elements w, x, y, z in a poset P are an *instance of* $(3+1)$ if we have that $x < y < z$ and w is incomparable to all of x, y, z. If P contains no instance of $(3+1)$, we say that P *avoids* $(3+1)$.

Call a poset P *weakly graded* if there exists a rank function $\mathrm{rk} : P \to \mathbf{N}$ such that if $a \lessdot b$ then $\mathrm{rk}(b) - \mathrm{rk}(a) = 1$ and such that the minimal occurring rank in each connected component is 0. Call a weakly graded poset *strongly graded* if all minimal elements are on the same rank and all maximal elements are on the same rank.[2] (Equivalently, a poset is

[2]We avoid the use of the unmodified word "graded" in the statement of theorems and results because of an ambiguity in the literature: some sources (e.g., [11]) use the word "graded" to mean "strongly graded," while many others (e.g., [7]) use "graded" to mean "weakly graded." Such is life; we hope the reader does

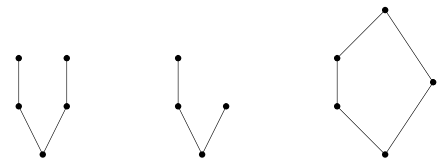


Figure 1: Three posets: the first is strongly graded, the second is weakly graded but not strongly graded, and the third is not weakly graded.

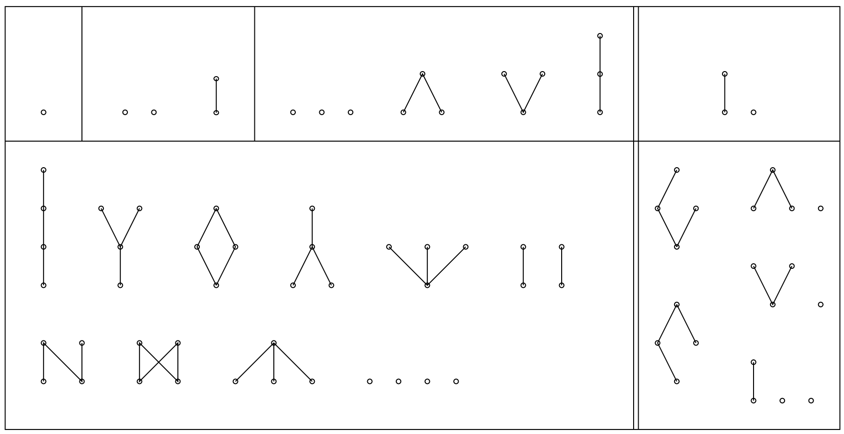


Figure 2: All weakly graded $(3+1)$-avoiding posets on four or fewer vertices. The doubled line separates the strongly graded posets from the others.

strongly graded if all maximal chains in the poset have the same length; in this case the rank function rk may be recovered by setting $\mathrm{rk}(v)$ to be the length of the longest chain whose maximal element is v.) Figure 1 gives examples of posets with these properties. The *height* of a weakly graded poset P is the number of vertices in the longest chain in P.

A weakly graded poset P of height $k+1$ has *vertex levels* $P(0)$, $P(1)$, ..., $P(k)$, where $P(i) = \{v \in P \mid \mathrm{rk}(v) = i\}$. If P is strongly graded, all the minimal elements are in $P(0)$ and all the maximal ones are in $P(k)$.

Figure 2 shows all unlabeled weakly graded $(3+1)$-avoiding posets on four or fewer vertices. Taking labelings into account, we see that for $n = 1, 2, 3$, and 4 the number of weakly graded $(3+1)$-avoiding posets on n vertices is 1, 3, 19, and 195, respectively. Of these, respectively 1, 3, 13 and 111 are strongly graded.

3 Local Conditions

In this section, we give a concise local condition that is equivalent to $(3+1)$-avoidance for weakly graded posets.

Given a weakly graded poset P, call a vertex $v \in P$ of rank i *up-seeing* if every vertex in $P(i+1)$ covers v. Similarly, call v *down-seeing* if v covers every vertex in $P(i-1)$. Let $\mathrm{V}(i)$ be the set of up-seeing vertices of rank i and let $\Lambda(i)$ be the set of all down-seeing vertices

not feel overburdened by the multiplication of adverbs.

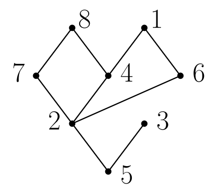


Figure 3: In the (weakly graded) poset pictured, the vertices labeled 1, 8, 4, 2 and 5 are up-seeing and the vertices labeled 2, 3 and 5 are down-seeing. The vertices labeled 6 and 7 are neither up- nor down-seeing.

of rank i. (As a mnemonic, think of v at the point of the V or Λ, with lots of edges going respectively up or down in the Hasse diagram of the poset.) These definitions are illustrated in Figure 3.

Theorem 3.1. *A weakly graded poset P is $(3+1)$-avoiding if and only if*

- *every vertex of P is up-seeing, down-seeing, or both, and*
- *every two vertices v, w such that $\mathrm{rk}(w) - \mathrm{rk}(v) \geq 2$ are comparable.*

Proof. Let P be a weakly graded $(3+1)$-avoiding poset. First, we show that two vertices whose ranks differ by 2 or more are comparable. Fix nonnegative integers i and j with $j - i \geq 2$, and choose a vertex v of rank i and a vertex w of rank j. Since P contains vertices of rank $j \geq i + 2$, it must contain a 3-chain $x \lessdot y \lessdot z$ such that $\mathrm{rk}(x) = i$. Since P avoids $(3+1)$, we must have that w is comparable to at least one of x, y, and z; since $\mathrm{rk}(w)$ is at least as large as $\mathrm{rk}(x)$, $\mathrm{rk}(y)$, and $\mathrm{rk}(z)$, we have in particular that $w > x$. Since $\mathrm{rk}(w) - \mathrm{rk}(x) \geq 2$, there exists $y' \in P$ such that $x < y' < w$. Finally, since P avoids $(3+1)$, we must have that v is comparable to at least one of x, y' and w; since $\mathrm{rk}(v)$ is no larger than $\mathrm{rk}(x)$, $\mathrm{rk}(y')$, and $\mathrm{rk}(w)$, we have in particular that $v < w$, as desired.

Second, we show that every vertex in P is up-seeing or down-seeing. Fix a nonnegative integer i and a vertex v of rank i, and suppose for contradiction that $v \notin \Lambda(i) \cup \mathrm{V}(i)$. Then there exist vertices u, w such that $\mathrm{rk}(u) = \mathrm{rk}(v) - 1$, $\mathrm{rk}(w) = \mathrm{rk}(v) + 1$, and v is incomparable to both u and w. But by the preceding paragraph, $u < w$, and so there is some vertex v' of rank i such that $u < v' < w$. This chain together with v is an instance of $(3+1)$ in P. This is a contradiction, so we must have $v \in \Lambda(i) \cup \mathrm{V}(i)$, as desired.

Now, we prove the converse: suppose P is a weakly graded poset such that every vertex is up-seeing or down-seeing and every two vertices v, w such that $\mathrm{rk}(w) - \mathrm{rk}(v) \geq 2$ are comparable; we will show P avoids $(3+1)$. Consider any 3-chain $x < y < z$ in P and any other vertex $w \in P$; we show that w is comparable to at least one of x, y, z. By the defining properties of P, if $\mathrm{rk}(w) < \mathrm{rk}(z) - 1$ then $w < z$ while if $\mathrm{rk}(w) > \mathrm{rk}(x) + 1$ then $w > x$, and in either case we have our result. The only remaining case is $\mathrm{rk}(z) - 1 = \mathrm{rk}(w) = \mathrm{rk}(x) + 1$. In this case, since w is either up- or down-seeing, we conclude that w is comparable to at least one of x and z. Thus, P avoids $(3+1)$, as desired. $\square$

Remark 1. We can weaken the second condition from inequality to equality and the result is still valid. This is immediate in the case of strongly graded posets and requires a brief argument (which we omit) in the case of weakly graded posets.

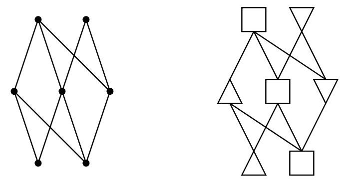


Figure 4: The Hasse diagram for the vigilant poset at left will be displayed as the image at right: all-seeing vertices are represented as squares, other vertices as triangles.

One consequence of Theorem 3.1 is that in our study of graded $(3+1)$-avoiding posets we need only consider posets in which every vertex is up-seeing or down-seeing. We make heavy use of this property in the following sections, so we give it a name: we say that a weakly graded poset P is *vigilant* if every vertex of P is up-seeing, down-seeing, or both. For similar reasons, we refer to vertices that are both up- and down-seeing as *all-seeing*.

We introduce the following convention for representing vigilant posets: vertices that are all-seeing are represented by squares, vertices that are up-seeing are represented by downwards-pointing triangles, and vertices that are down-seeing are represented by upwards-pointing triangles. (Thus, each vertex has horizontal edges on the sides on which it is connected to all vertices.) This convention is illustrated in Figure 4.

4 Simplifications

In this section, we introduce four operations that allow us to count vigilant posets by working instead with simpler objects. We show that $(3+1)$-avoidance will be mostly compatible with these simplifications, reducing the problem of enumerating graded $(3+1)$-avoiding posets basically to studying vigilant posets of height 2. In Section 4.1 we work with weakly graded posets, while in Sections 4.2 and 4.3 we restrict ourselves to strongly graded posets. (We will return to weakly graded posets in Section 8.)

4.1 Trimming

We call a vigilant poset P *trimmed* if it has the following properties:

- every rank has at most one all-seeing vertex,
- the all-seeing vertices are unlabeled, and
- the other m vertices are labeled with $[m]$.

Given a weakly graded poset P, there is a naturally associated trimmed poset, denoted $\mathrm{trim}(P)$, that we get by removing the all-seeing vertices from P, adding a single unlabeled all-seeing vertex to any vertex level from which we removed all-seeing vertices, and relabeling the other vertices so as to preserve the relative order of labels. Figure 5 provides one illustration of this operation.

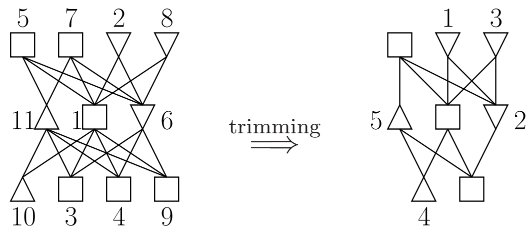


Figure 5: A strongly graded $(3+1)$-avoiding poset and the associated trimmed poset.

Proposition 4.1. *The weakly graded vigilant poset P avoids $(3+1)$ if and only if* $\operatorname{trim}(P)$ *does.*

Proof. It is routine to check that neither of the conditions of Theorem 3.1 is affected by the trimming map. ☐

Since we lose very little information when we replace the poset P by the trimmed poset $\operatorname{trim}(P)$, Proposition 4.1 suggests that we can reduce the enumeration of labeled graded $(3+1)$-avoiding posets to the enumeration of trimmed $(3+1)$-avoiding posets. The following proposition makes this intuition precise.

Proposition 4.2. *Let w_n be the number of weakly graded $(3+1)$-avoiding posets on n vertices and let*

$$W(x) = \sum_n w_n \frac{x^n}{n!}$$

be the exponential generating function for labeled weakly graded $(3+1)$-avoiding posets. Let $a_{n,r}$ be the number of trimmed $(3+1)$-avoiding posets with r all-seeing vertices and n other vertices and let

$$W_T(x,z) = \sum_{n,r} a_{n,r} \frac{x^n}{n!} z^r$$

be the generating function for trimmed $(3+1)$-avoiding posets, exponential in x and ordinary in z. Then

$$W(x) = W_T(x, e^x - 1).$$

The same result holds if we restrict attention to the strongly graded posets.

Proof. Let $\mathcal{P}$ be the class of weakly graded $(3+1)$-avoiding posets, $\mathcal{T}$ be the class of trimmed $(3+1)$-avoiding posets, and $\mathcal{A}$ be the class of nonempty antichains. Following the ideas of Appendix A, we apply Lemma A.2: every member $P \in \mathcal{P}$ arises uniquely by replacing each all-seeing vertex of $\operatorname{trim} P \in \mathcal{T}$ by a nonempty antichain and distributing labels. Thus $\mathcal{P} = \mathcal{T} \circ \mathcal{A}$ and we have our result. (The application of Lemma A.2 is justified, despite a minor difference between this setting and that of the lemma, by Lemma A.3.) ☐

4.2 Layering

Suppose we have two trimmed strongly graded posets P_1 and P_2 of heights a and b, respectively. We can *layer* P_1 and P_2 by letting the lowest-ranked elements in P_2 cover all

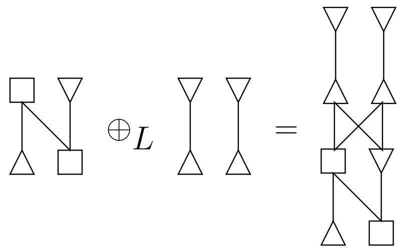


Figure 6: Layering two L-indecomposable posets to form a new poset. (Labels are suppressed for readability.)

highest-ranked elements in P_1 and relabeling in a way consistent with the labelings of P_1 and P_2. (Thus, there are many ways to layer P_1 and P_2; all the resulting posets are isomorphic.) We denote the resulting poset of height $a+b$ by $P_1 \oplus_L P_2$. See for example Figure 6. This operation is also called the *ordinal sum* of the posets P_1 and P_2. In the context of vigilant posets, it is an especially nice operation because a vertex in P_1 or P_2 which is up-seeing and/or down-seeing retains that property in $P_1 \oplus_L P_2$.

Call a nonempty strongly graded trimmed poset P with height $k \geq 1$ *L-indecomposable* if P is trimmed and there is no $i < k-1$ for which every vertex in $P(i)$ is up-seeing (equivalently, there is no $i > 0$ for which every vertex in $P(i)$ is down-seeing). This word choice is motivated by the existence of a decomposition of trimmed posets into L-indecomposables.

Proposition 4.3. *A trimmed strongly graded poset P can be written uniquely as*

$$P = P_1 \oplus_L P_2 \oplus_L \cdots \oplus_L P_k,$$

for a sequence $(P_1, P_2, \ldots, P_k)$ of L-indecomposable posets.

Proof. Take the smallest rank i for which $P(i)$ has all up-seeing vertices. If $i = k$, then P is L-indecomposable. Otherwise, we can write $P = P_1 \oplus_L P'$, where P_1 has height $i+1$ and is L-indecomposable by the minimality of i. Repeating this process gives us the desired sequence, which is obviously unique. □

Proposition 4.4. *If a trimmed strongly graded poset P decomposes into L-indecomposables as $P = P_1 \oplus_L \cdots \oplus_L P_k$, then P avoids $(3+1)$ if and only if all of the P_i avoid $(3+1)$.*

Proof. One direction is trivial: if any of the P_i contains an instance of $(3+1)$ then certainly P does as well. For the other direction, suppose that all the P_i avoid $(3+1)$; we will show that P also avoids $(3+1)$. It suffices to check that P satisfies the local conditions in Theorem 3.1. The first condition, that every vertex is up-seeing or down-seeing or both, is satisfied by construction and by the fact that the P_i have this property. Thus, we are left to check the second condition, that every vertex is comparable to all vertices two ranks above it. Note that we also don't have to check anything for P_i that have height 1, as the desired condition is automatically satisfied when we look at a pair of vertices with ranks 2 apart that either involve or straddle a layer of height 1. As a final simplification, observe that since the P_i avoid $(3+1)$ they satisfy our condition, so we only have to check vertices in the top and

next-to-top vertex levels of each of the P_i. Thus, it's clear that we can restrict to the case $P = P_1 \oplus_L P_2$.

Let a be the height of P_1. First, choose any $v \in P(a-1) = P_1(a-1)$ and any $w \in P(a+1) = P_2(1)$. Since P_2 is strongly graded, there is some $w' \in P_2(0) = P(a)$ such that $w' \lessdot w$. By construction, every vertex in $P(a)$ is down-seeing, so $v \lessdot w'$ and thus $v < w$ as desired.

Second, choose any $v \in P(a-2) = P_1(a-2)$ and any $w \in P(a) = P_2(0)$. Since P_1 is strongly graded, there is some $v' \in P_1(a-1) = P(a-1)$ such that $v \lessdot v'$. By construction, $v' \lessdot w$ and thus $v < w$ in this case as well. This completes the proof. □

Propositions 4.3 and 4.4 simplify the problem of counting strongly graded (3+1)-avoiding posets: it now suffices to count L-indecomposable posets and then layer them together. As we will see in Proposition 6.6, this is a simple task with generating functions. Thus, we now turn our attention to enumerating L-indecomposable $(3+1)$-avoiding posets.

4.3 Sticking and Gluing

In order to enumerate L-indecomposable posets, we break them down into more manageable pieces.[3] In particular, we introduce two associative operations that can be used to build every L-indecomposable poset. Suppose that we have L-indecomposable posets P_1 and P_2 of height a and b, respectively. If P_1 has no all-seeing vertex of top rank and P_2 has no all-seeing vertex of bottom rank, then we allow the following two constructions.

- We can *stick* P_1 and P_2 to form a new poset $P = P_1 \oplus_S P_2$ of height $a + b - 1$, as follows:
  - The vertex set of P is the disjoint union of the vertex sets of P_1 and P_2.
  - For $i = 1, 2$, if $v, w \in P_i$, then $v < w$ in P if and only if $v < w$ in P_i.
  - If $v \in P_1$ and $w \in P_2$ then $v < w$ in P unless $\mathrm{rk}(v) = a - 1$ and $\mathrm{rk}(w) = 0$. In this case, v and w are incomparable.
  - We distribute labels to vertices of P consistent with the labelings of P_1 and P_2.
- We can *glue* P_1 and P_2 to form a new poset $P = P_1 \oplus_G P_2$ of height $a+b-1$, as follows:
  - The vertex set of P is the disjoint union of the following three sets: the vertex set of P_1, the vertex set of P_2, and a singleton set $\{\Xi\}$.
  - For $i = 1, 2$, if $v, w \in P_i$ then $v < w$ in P if and only if $v < w$ in P_i.
  - If $v \in P_1$ and $w \in P_2$ then $v < w$ in P unless $\mathrm{rk}(v) = a - 1$ and $\mathrm{rk}(w) = 0$. In this case, we set v and w to be incomparable.
  - If $v \in P_1$ is not of top rank then $v < \Xi$ in P. If instead $\mathrm{rk}(v) = a - 1$ then v and Ξ are incomparable.

[3]In defense of what seems like a bad joke, the original meaning of the word "atom" was "indecomposable," but subatomic particles stubbornly exist.

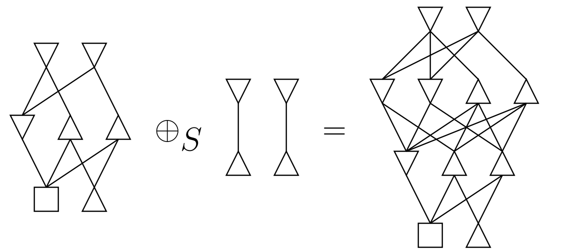


Figure 7: An example of sticking two L-indecomposable posets. (Labels are suppressed for readability.)

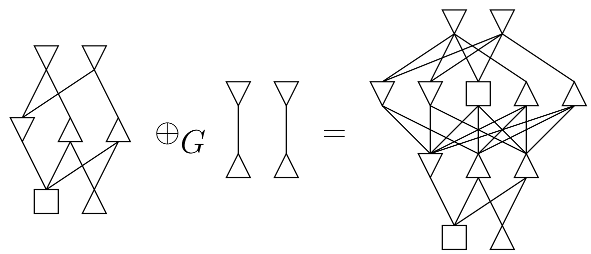


Figure 8: An example of gluing two L-indecomposable posets. (Labels are suppressed for readability.)

- If $w \in P_2$ is not of bottom rank then $\Xi < w$ in P. If instead $\mathrm{rk}(w) = 0$ then w and Ξ are incomparable.
- We distribute labels to vertices of P consistent with the labelings of P_1 and P_2.

Note that gluing is basically sticking, except we add an all-seeing vertex to the boundary rank. Furthermore, as in the case of layering, a vertex in P_1 or P_2 that it up-seeing or down-seeing keeps this status after either gluing or sticking. Figure 7 shows an example of sticking two posets; Figure 8 shows a gluing of two posets.

In the context of L-indecomposable posets, these are good operations since they preserve L-indecomposability, as the next result shows.

Proposition 4.5. *Suppose P_1 and P_2 are L-indecomposable posets such that P_1 has no all-seeing vertices of top rank and P_2 has no all-seeing vertices of bottom rank. The posets $P_1 \oplus_S P_2$ and $P_1 \oplus_G P_2$ are L-indecomposable.*

Proof. We show that $P = P_1 \oplus_S P_2$ is L-indecomposable; the proof for gluing is essentially identical. Let P_1 have height $a + 1$ and let P_2 have height $b + 1$. For any i such that $0 < i < a$ we have $P(i) = P_1(i)$. Since P_1 is L-indecomposable, $P_1(i)$ contains both vertices that are up-but-not-down-seeing and vertices that are down-but-not-up-seeing. Similarly, for $a < i < a + b$ we have $P(i) = P_2(i - a)$ contains both types of vertices. So it remains to check that $P(a) = P_1(a) \cup P_2(0)$ contains both types of vertices. Indeed, since P_1 is L-indecomposable we have that $P_1(a)$ contains some vertices that are not up-seeing, and since P_2 is L-indecomposable we have that $P_2(0)$ contains some vertices that are not down-seeing. Thus, no vertex level of $P = P_1 \oplus_S P_2$ has all up-seeing nor all down-seeing vertices and so P is L-indecomposable, as desired. $\square$

The key observation of this section is that any L-indecomposable poset can be decomposed at any rank by exactly one of the two operations we've just defined.

Proposition 4.6. *Let P be an L-indecomposable poset of height k, $k \geq 3$. For any rank i, $0 < i < k-1$, exactly one of the following is true:*

- *there exist L-indecomposable posets P_1 of height $i+1$ and P_2 of height $k-i$ such that $P = P_1 \oplus_S P_2$, or*
- *there exist L-indecomposable posets P_1 of height $i+1$ and P_2 of height $k-i$ such that $P = P_1 \oplus_G P_2$.*

Furthermore, P_1 and P_2 are uniquely determined by i.

Proof. The vertex set of P is the union of the following sets:

- $P(> i)$, the set of vertices of rank greater than i,
- $P(< i)$, the set of vertices of rank less than i,
- $A(i)$, the set that contains the unique all-seeing vertex on rank i if it exists and is empty otherwise,
- $U(i)$, the other up-seeing vertices on rank i, and
- $D(i)$, the other down-seeing vertices on rank i.

The first three sets may be either empty or nonempty; the last two sets are non-empty because P is L-indecomposable. Now, let P_1 be the subposet of P induced by $P(< i) \cup U(i)$ and let P_2 be the subposet induced by $P(> i) \cup D(i)$. These are two smaller L-indecomposable posets. It is easy to see that if $A(i)$ is empty then P_1 and P_2 stick to form P while if $A(i)$ is nonempty then P_1 and P_2 glue to form P, and that this is also the only way to decompose P using gluing or sticking into P_1 and P_2 such that P_1 has height $i+1$. $\square$

Corollary 4.7. *For $k \geq 1$, every L-indecomposable poset P of height $k+1$ can be written uniquely in the form*

$$P = P_1 \oplus_{\alpha_1} P_2 \oplus_{\alpha_2} \cdots \oplus_{\alpha_{k-1}} P_k,$$

where

- *each α_i is one of S and G,*
- *each P_i is an L-indecomposable poset of height exactly 2, and*
- *no elements in any P_i are all-seeing, except possibly a single vertex in each of $P_1(0) = P(0)$ and $P_k(1) = P(k)$.*

Moreover, if $P_1, \ldots, P_k$ satisfy the above conditions, then the poset

$$P_1 \oplus_{\alpha_1} P_2 \oplus_{\alpha_2} \cdots \oplus_{\alpha_{k-1}} P_k$$

is L-indecomposable.

We call $P_1, \ldots, P_k$ the *quarks* of P. We'll frequently refer to P_1 and P_k as the *bottom quark* and *top quark* of P, respectively. Thus quarks are height-2 L-indecomposable posets with no all-seeing vertices, except possibly the top and bottom quarks, which may have one all-seeing vertex. Corollary 4.7 tells us that an L-indecomposable poset P of height $k+1$ has exactly k quarks $P_1, \ldots, P_k$, where P_i is exactly the subposet induced by the vertices in $\mathrm{V}(i+1) \cup \Lambda(i)$ that are not all-seeing.

Now we can connect our characterization of L-indecomposable posets as quarks that have been glued or stuck together to our ultimate goal of studying $(3+1)$-avoiding posets.

Proposition 4.8. *For two L-indecomposable posets P_1 and P_2 such that P_1 has no all-seeing vertex of top rank and P_2 has no all-seeing vertex of bottom rank,*

1. *$P_1 \oplus_G P_2$ is $(3+1)$-avoiding if and only if both P_1 and P_2 are, and*
2. *$P_1 \oplus_S P_2$ is $(3+1)$-avoiding if and only if the following hold:*
   - *both P_1 and P_2 are $(3+1)$-avoiding, and*
   - *if Q_1 is the top quark of P_1 and Q_2 is the bottom quark of P_2 then Q_1 has no isolated vertices on its bottom rank or Q_2 has no isolated vertices on the top rank (or both).*

Proof. We can combine P_1 and P_2 as either $P_1 \oplus_G P_2$ or $P_1 \oplus_S P_2$. Let P be result in either case.

In both cases, one direction is clear: since P contains P_1 and P_2 as induced subposets, if P avoids $(3+1)$ then P_1 and P_2 do as well. We now show that the other direction also holds, except in the mentioned special case.

Assume P_1 and P_2 avoid $(3+1)$. As before, Theorem 3.1 tells us that P avoids $(3+1)$ if and only if every pair of vertices $v, w \in P$ such that $\mathrm{rk}(w) - \mathrm{rk}(v) = 2$ also satisfies $v < w$. Since P_1 and P_2 are $(3+1)$-avoiding, it suffices to check only the case $v \in P_1$ and $w \in P_2$. Let P_1 have height $a+1$, so the boundary rank in P is $P(a)$. There are three possible cases: $\mathrm{rk}(v) = a-2$, $\mathrm{rk}(v) = a-1$ and $\mathrm{rk}(v) = a$. If $\mathrm{rk}(v) = a-2$, then w, being comparable to every vertex in $P(a-1)$, must be comparable to v as well, as desired. A similar argument takes care of the case $\mathrm{rk}(v) = a$. The only remaining case is $\mathrm{rk}(v) = a-1$ and $\mathrm{rk}(w) = a+1$.

Now we consider gluing and sticking separately. Suppose $P = P_1 \oplus_G P_2$. By construction, P contains an all-seeing vertex, say u, of rank a. Thus $v < u < w$ and so P always avoids $(3+1)$ in this case, as desired.

Suppose $P = P_1 \oplus_S P_2$. If v is up-seeing or if w is down-seeing we're done, so it suffices to consider the case that v is down-seeing and w is up-seeing, or equivalently, when v is on the bottom rank of the top quark Q_1 of P_1 and w is on the top rank of the bottom quark Q_2 of P_2. If v is not isolated in Q_1, then v must be connected to at least one vertex on the top rank of Q_1; this vertex is up-seeing, so $v < w$. Similarly, we have $v < w$ whenever w is not isolated in Q_2. Finally, if v is isolated in Q_1 and w is isolated in Q_2 then it's easy to check that there is no $u \in P(a)$ such that $v < u < w$. Thus P avoids $(3+1)$ if and only if there are not isolated vertices in both the bottom rank of Q_1 and the top rank of Q_2, as claimed. $\square$

The punchline of this section is that we now have a complete characterization of L-indecomposable $(3+1)$-avoiding posets.

Corollary 4.9. *An L-indecomposable poset P is $(3+1)$-avoiding if and only if the decomposition $P = P_1 \oplus_{\alpha_1} P_2 \oplus_{\alpha_2} \cdots \oplus_{\alpha_{k-1}} P_k$ into quarks satisfies the following condition: for every occurrence of $P_i \oplus_S P_{i+1}$ in the decomposition, either P_i has no isolated vertices on its bottom level or P_{i+1} has no isolated vertices on its top level or both.*

5 Quarks

Corollary 4.9 implies that studying L-indecomposable $(3+1)$-avoiding posets reduces to studying quarks, which (except for possibly the top and bottom quarks) are height-2 labeled posets with no all-seeing vertices. A small but useful observation is that such a height-2 labeled poset P with m vertices in $P(0)$ and n vertices in $P(1)$ is, up to differences in the labeling scheme, just a bipartite graph on the disjoint union $[m] \uplus [n]$. In this section, we set out to enumerate quarks by enumerating such graphs, keeping track of some simple structural information about them.

We define a family of sets $A_\mu^\nu(m,n)$, where μ and ν are subsets (possibly empty) of $\{\square, \circ, \boxtimes, \otimes\}$, as follows:

- $A_\mu^\nu(m,n)$ is the set of bipartite graphs on $[m] \uplus [n]$ with some restrictions. The elements of ν correspond to restrictions on the vertices in $[n]$ and the elements of μ correspond to restrictions on the vertices of $[m]$. (Here the placement of indices is meant to suggest that vertices in $[m]$ form a bottom level and the vertices in $[n]$ a top level.) An empty set of symbols corresponds to no restrictions on the corresponding set.

- A $\square$ corresponds to the requirement that there be at least one all-seeing vertex; a $\boxtimes$ corresponds to the requirement that there be no all-seeing vertex.

- A $\circ$ corresponds to the requirement that there be an isolated vertex; a $\otimes$ corresponds to the requirement that there be no isolated vertex.

For example, $A(m,n)$ is the set of all bipartite graphs on $[m] \uplus [n]$ and $A_\boxtimes^\square(m,n)$ is the subset of $A(m,n)$ containing those graphs with at least one all-seeing vertex in $[n]$ but no all-seeing vertices in $[m]$. Note that some collections of these restrictions allow no legal graphs: we have $A_\circ^\square(m,n) = \varnothing$ for all m and n because we cannot have both an isolated vertex in $[m]$ and an all-seeing vertex in $[n]$, while $A^{\circ\otimes}(m,n) = \varnothing$ because we cannot both enforce and prohibit an isolated vertex in $[n]$.

We are particularly interested in quarks, which, roughly speaking, are those graphs with no all-seeing vertices; thus, for $\nu, \mu \subset \{\circ, \otimes\}$ we define $B_\mu^\nu(m,n) = A_{\{\boxtimes\}\cup\mu}^{\{\boxtimes\}\cup\nu}(m,n)$. For example, $B_\circ^\otimes(m,n)$ is the set of bipartite graphs on $[m] \uplus [n]$ with no all-seeing vertices, no isolated vertices in $[n]$, and at least one isolated vertex in $[m]$. For each B_μ^ν, let

$$F_\mu^\nu(x) = \sum_{m,n\geq 1} |B_\mu^\nu(m,n)| \frac{x^{m+n}}{m!n!} \tag{1}$$

be the corresponding generating function. Finally, let B_μ^ν be the union over m and n of all $B_\mu^\nu(m,n)$. Note that we have a disjoint union

$$B = B_\circ^\circ \cup B_\otimes^\circ \cup B_\circ^\otimes \cup B_\otimes^\otimes,$$

which manifests as a sum of formal power series

$$F = F_{\circ}^{\circ} + F_{\otimes}^{\circ} + F_{\circ}^{\otimes} + F_{\otimes}^{\otimes}.$$

Proposition 5.1. *Let*

$$\Psi(x) = \sum_{m,n\geq 0} \frac{2^{mn}x^{m+n}}{m!n!}$$

and let F_μ^ν be defined as in Equation (1). *We have*

$$F_{\circ}^{\circ}(x) = (1-e^{-x})^2\Psi(x),$$
$$F_{\otimes}^{\circ}(x) = F_{\circ}^{\otimes}(x) = (1-e^{-x})((2e^{-x}-1)\Psi(x)-1),$$

and

$$F_{\otimes}^{\otimes}(x) = (2e^{-x}-1)((2e^{-x}-1)\Psi(x)-1).$$

Proof. See Appendix B. □

6 Strongly Graded Posets

In this section, we use the F_μ^ν as building blocks to obtain the generating function for L-indecomposable $(3+1)$-avoiding posets, and then proceed to enumerate all strongly graded $(3+1)$-avoiding posets. We begin by encoding an L-indecomposable poset in terms of a *word* that keeps track of its quarks and how they are combined (i.e., gluing and sticking). Then we use the transfer-matrix method to enumerate words while keeping track of the restrictions imposed by Corollary 4.9.

For a quark with no all-seeing vertices (i.e., a quark in B), we define its *type* to be the symbol B_μ^ν, corresponding to the unique subset among the four B_μ^ν to which it belongs. (This is a slight abuse of notation that will always be unambiguous in context.) Now, define a *word* to be any monomial in the noncommutative algebra $\mathbf{R}\langle\langle S, G, B_{\circ}^{\circ}, B_{\otimes}^{\circ}, B_{\circ}^{\otimes}, B_{\otimes}^{\otimes}\rangle\rangle$. We now encode the properties of being L-indecomposable and $(3+1)$-avoiding into conditions on words.

Definition 6.1. We say that a word L is *legal* if for some $k \geq 1$ there are $\alpha_i \in \{S, G\}$ and $B_i \in \{B_{\circ}^{\circ}, B_{\otimes}^{\circ}, B_{\circ}^{\otimes}, B_{\otimes}^{\otimes}\}$ such that $L = \alpha_0 B_1 \alpha_1 B_2 \alpha_2 \cdots B_{k-1}\alpha_{k-1}B_k\alpha_k$, and none of the following occur:

1. $\alpha_0 = S$ and B_1 has a $\circ$ in the superscript;
2. $\alpha_k = S$ and B_k has a $\circ$ in the subscript;
3. there is some i, $1 \leq i \leq k-1$, such that B_i has a $\circ$ in the subscript, $\alpha_i = S$, and B_{i+1} has a $\circ$ in the superscript.

We define a *weight function* $\mathrm{wt} : \mathbf{R}\langle\langle S, G, B_{\circ}^{\circ}, B_{\otimes}^{\circ}, B_{\circ}^{\otimes}, B_{\otimes}^{\otimes}\rangle\rangle \to \mathbf{R}[\![x, z]\!]$ as follows: we set $\mathrm{wt}(S) = 1$, $\mathrm{wt}(G) = z$, and $\mathrm{wt}(B_\mu^\nu) = F_\mu^\nu$ and we extend by linearity and multiplication.

Theorem 6.2. *Let $I_{\geq 2}(x,z)$ be the generating function for L-indecomposable* $(3+1)$*-avoiding posets of height at least* 2*, where the variable z counts all-seeing vertices, the variable x counts other vertices, and $I_{\geq 2}(x,z)$ is exponential in x and ordinary in z. Then*

$$I_{\geq 2}(x,z) = \sum_{L} \mathrm{wt}(L),$$

where the sum is over all legal words L.

Proof. Let P be an L-indecomposable $(3+1)$-avoiding poset. Suppose P decomposes into quarks as $P = P_1 \oplus_{\alpha_1} \cdots \oplus_{\alpha_{k-1}} P_k$ and set $W(P)$ to be the word $\alpha_0 B_1 \alpha_1 B_2 \alpha_2 \cdots B_{k-1} \alpha_{k-1} B_k \alpha_k$ defined as follows:

- for $1 < i < k$, B_i is the type of P_i;
- if P has no all-seeing vertex of top rank, we set $\alpha_k = S$ and B_k to be the type of P_k; otherwise, we set $\alpha_k = G$ and set B_k to be the type of P_k with the all-seeing vertex removed; and
- if P has no all-seeing vertex of rank 0, we set $\alpha_0 = S$ and B_1 to be the type of P_1; otherwise, we set $\alpha_0 = G$ and set B_1 to be the type of P_1 with the all-seeing vertex removed.

Note that it is not *a priori* clear that the B_i are well-defined: height-2 L-indecomposable posets only have a type if they have no all-seeing vertices. However, in our case the B_i really are well-defined by Corollary 4.7. In fact, it's easy to check that the constraints imposed on the P_i and α_i by Corollary 4.7 correspond precisely to the condition that $W(P)$ is a legal word. Given a legal word L, we now show that the generating function for posets P such that $W(P) = L$ is precisely $\mathrm{wt}(L)$; summing over all legal words L, our result will follow immediately.

Fix a word $L = \alpha_0 B_1 \alpha_1 B_2 \alpha_2 \cdots B_{k-1} \alpha_{k-1} B_k \alpha_k$, and consider its preimage $W^{-1}(L) = \{P \mid W(P) = L\}$. Any $P \in W^{-1}(L)$ can be written in the form $P = P_1 \oplus_{\alpha_1} \cdots \oplus_{\alpha_{k-1}} P_k$ with the types of the P_i (or in the case that P_1 has an all-seeing vertex v, the type of $P_1 \setminus \{v\}$, and similarly for P_k) determined by the B_i. However, after we fix the type B_i, any quark of that type can be used as part of an L-indecomposable $(3+1)$-avoiding poset. Thus, the posets in the preimage of L contribute exactly F_μ^ν for each occurrence of $B_i = B_\mu^\nu$. Furthermore, each occurrence of $\alpha_i = G$ corresponds exactly to a single all-seeing vertex, and so contributes z. Thus, applying Lemma A.1, the generating function for posets in $W^{-1}(L)$ is exactly $\mathrm{wt}(L)$. It follows that $I_{\geq 2}(x,z)$ is exactly the result of summing $\mathrm{wt}(L)$ over the legal words L, as desired. □

Corollary 6.3. *The generating function for all nonempty L-indecomposable* $(3+1)$*-avoiding posets is*

$$I(x,z) = z + \sum_{L} \mathrm{wt}(L),$$

where the sum is over all legal words L.

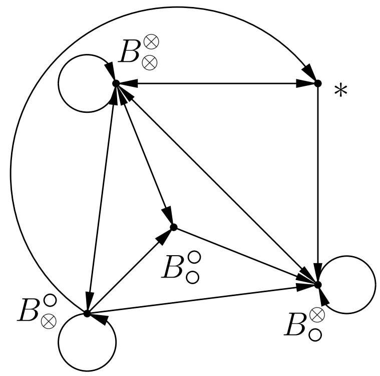


Figure 9: The S-labeled edges of the graph G_w defined in the proof of Proposition 6.4. Each pair of vertices is also joined by directed edges labeled G (not shown).

The preceding results establish that to enumerate posets we may focus our energies on enumerating words. We accomplish this task with the transfer-matrix method.

Proposition 6.4. *Let M_W be the matrix*

$$M_W = G \cdot \begin{bmatrix} B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \\ B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \\ B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \\ B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \end{bmatrix} + S \cdot \begin{bmatrix} 0 & B^{\bigcirc}_{\otimes} & 0 & B^{\otimes}_{\otimes} \\ 0 & B^{\bigcirc}_{\otimes} & 0 & B^{\otimes}_{\otimes} \\ B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \\ B^{\bigcirc}_{\bigcirc} & B^{\bigcirc}_{\otimes} & B^{\otimes}_{\bigcirc} & B^{\otimes}_{\otimes} \end{bmatrix}$$

with entries in the noncommutative algebra $\mathbf{R}\langle\langle S, G, B^{\bigcirc}_{\bigcirc}, B^{\bigcirc}_{\otimes}, B^{\otimes}_{\bigcirc}, B^{\otimes}_{\otimes}\rangle\rangle$ of words. The sum of the legal words of length $2k+1$ is

$$\begin{bmatrix} G \cdot B^{\bigcirc}_{\bigcirc} & (S+G)B^{\bigcirc}_{\otimes} & G \cdot B^{\otimes}_{\bigcirc} & (S+G)B^{\otimes}_{\otimes} \end{bmatrix} \cdot (M_W)^{k-1} \cdot \begin{bmatrix} G \\ G \\ S+G \\ S+G \end{bmatrix}.$$

Proof. Consider the graph G_w with vertices $\{*, B^{\bigcirc}_{\bigcirc}, B^{\bigcirc}_{\otimes}, B^{\otimes}_{\bigcirc}, B^{\otimes}_{\otimes}\}$ and the following directed, labeled edges: for each pair u, v of vertices (allowing $u = v$), G_w has a directed edge $u \xrightarrow{S} v$ unless $u = B^{\nu}_{\bigcirc}$ or $u = *$ and $v = B^{\bigcirc}_{\mu}$ or $v = *$, and for every pair u, v of vertices, G_w has a directed edge $v \xrightarrow{G} w$. The graph G_w is illustrated in Figure 9.

We identify each walk

$$* \xrightarrow{\alpha_0} B_1 \xrightarrow{\alpha_1} \cdots B_k \xrightarrow{\alpha_k} *$$

with the word

$$\alpha_0 B_1 \alpha_1 \cdots B_k \alpha_k.$$

Observe that the first two conditions in Definition 6.1 exactly correspond to the restrictions on edges involving $*$ and the final condition exactly corresponds to edges not involving $*$. Thus the legal words are exactly the walks on this graph that start and end at $*$, with no intermediate instances of $*$. We enumerate these walks using the transfer-matrix method, as in [10, Section 4.7].

Let $X = \begin{bmatrix} B^{\circ}_{\circ} & 0 & 0 & 0 \\ 0 & B^{\circ}_{\otimes} & 0 & 0 \\ 0 & 0 & B^{\otimes}_{\circ} & 0 \\ 0 & 0 & 0 & B^{\otimes}_{\otimes} \end{bmatrix}$ and $Y = G \cdot \mathbb{I} + S \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Examining G_w and applying [10, Theorem 4.7.1], we have that the sum of the words associated to the aforementioned walks is

$$\begin{bmatrix} G\cdot & (S+G) & G & (S+G) \end{bmatrix} XYXY \cdots X \begin{bmatrix} G \\ G \\ S+G \\ S+G \end{bmatrix},$$

which is equivalent to the desired expression. $\square$

Corollary 6.5. *Let M be the matrix*

$$M = z \cdot \begin{bmatrix} F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \\ F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \\ F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \\ F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \end{bmatrix} + \begin{bmatrix} 0 & F^{\circ}_{\otimes} & 0 & F^{\otimes}_{\otimes} \\ 0 & F^{\circ}_{\otimes} & 0 & F^{\otimes}_{\otimes} \\ F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \\ F^{\circ}_{\circ} & F^{\circ}_{\otimes} & F^{\otimes}_{\circ} & F^{\otimes}_{\otimes} \end{bmatrix}$$

(whose entries are in $\mathbf{R}[\![x, z]\!]$). For $k \geq 1$, the generating function for L-indecomposable $(3+1)$-avoiding posets of height $k+1$ is

$$I_{k+1}(x, z) = \begin{bmatrix} zF^{\circ}_{\circ} & (1+z)F^{\circ}_{\otimes} & zF^{\otimes}_{\circ} & (1+z)F^{\otimes}_{\otimes} \end{bmatrix} \cdot M^{k-1} \cdot \begin{bmatrix} z \\ z \\ 1+z \\ 1+z \end{bmatrix},$$

and the generating function for all L-indecomposable $(3+1)$-avoiding posets of height at least 2 is

$$I_{\geq 2}(x, z) = \begin{bmatrix} zF^{\circ}_{\circ} & (1+z)F^{\circ}_{\otimes} & zF^{\otimes}_{\circ} & (1+z)F^{\otimes}_{\otimes} \end{bmatrix} \cdot (\mathbb{I} - M)^{-1} \cdot \begin{bmatrix} z \\ z \\ 1+z \\ 1+z \end{bmatrix}.$$

Proof. The result follows from Theorem 6.2 and the fact that the weight map wt is an algebra homomorphism between $\mathbf{R}\langle\!\langle S, G, B^{\circ}_{\circ}, B^{\circ}_{\otimes}, B^{\otimes}_{\circ}, B^{\otimes}_{\otimes} \rangle\!\rangle$ and $\mathbf{R}[\![x, z]\!]$. $\square$

Now that we have enumerated L-indecomposable $(3+1)$-avoiding posets, the only remaining step is to express the generating function for all $(3+1)$-avoiding posets in terms of the generating function for L-indecomposables. This turns out to be extremely simple.

Proposition 6.6. *Let $I(x, z)$ be the generating function of nonempty L-indecomposable $(3+1)$-avoiding posets. Then the generating function $G_T(x, z)$ for all trimmed strongly graded $(3+1)$-avoiding posets is given by*

$$G_T(x, z) = (1 - I(x, z))^{-1}.$$

Proof. By Proposition 4.3 and Proposition 4.4 each trimmed $(3+1)$-avoiding poset P corresponds to a unique sequence $P_1 \oplus_L P_2 \oplus_L \cdots \oplus_L P_k$ of L-indecomposable $(3+1)$-avoiding posets, and all such sequences give a trimmed $(3+1)$-avoiding poset P. Now we apply Lemma A.2: if $\mathcal{C}$ is the class of chains, $\mathcal{I}$ is the class of L-indecomposable $(3+1)$-avoiding posets and $\mathcal{T}$ is the class of trimmed $(3+1)$-avoiding posets then we have $\mathcal{T} = \mathcal{C} \circ \mathcal{I}$. The generating function for $\mathcal{C}$ is $\sum_{n \geq 0} n! \cdot \frac{x^n}{n!} = (1-x)^{-1}$, so we have our result. □

The only thing remaining is arithmetic.

Theorem 6.7. *The generating function for all strongly graded* $(3+1)$*-avoiding posets is*

$$1 + \frac{e^{2x}(2e^x - 3) + e^x(e^x - 2)^2 \Psi(x)}{e^x(2e^x + 1) + (e^{2x} - 2e^x - 1)\Psi(x)}.$$

Proof. This is just a calculation, combining Corollaries 6.3 and 6.5 with Propositions 4.2 and 6.6. See Appendix C for Mathematica code that performs the necessary computations. For $\#P = 0, 1, \ldots,$ the resulting number of posets is 1, 1, 3, 13, 111, 1381, 22383, ... □

7 Strongly Graded Posets Counted by Height

In this section, we refine the generating function of the previous sections to count strongly graded $(3+1)$-avoiding posets with n vertices of height k. (This refinement is a natural one to ask for on its own terms; it will also be of use to us when we enumerate weakly graded $(3+1)$-avoiding posets in the next section.) The only change in our approach is that we keep track of the height of the poset as we glue and stick quarks, and then again as we layer L-indecomposables. To this end, let $b_{n,k}$ be the number of strongly graded $(3+1)$-avoiding posets on n vertices of height k and let

$$H(x,t) = \sum_{n,k} b_{n,k} \frac{x^n}{n!} t^k \tag{2}$$

be the generating function for these numbers. To compute $H(x,t)$, we return to the ideas of Section 6. It follows directly from Corollary 6.5 that the generating function for L-indecomposable strongly graded $(3+1)$-avoiding posets of height 2 or more is

$$H_I(x,z,t) = t^2 \begin{bmatrix} zF_{\bigcirc}^{\bigcirc} & (1+z)F_{\otimes}^{\bigcirc} & zF_{\bigcirc}^{\otimes} & (1+z)F_{\otimes}^{\otimes} \end{bmatrix} \cdot (\mathbb{I} - t \cdot M)^{-1} \cdot \begin{bmatrix} z \\ z \\ 1+z \\ 1+z \end{bmatrix}.$$

If we let $H_T(x,z,t)$ be the generating function for trimmed strongly graded $(3+1)$-avoiding posets, then we have from Proposition 6.6 that

$$H_T(x,z,t) = (1 - tz - H_I(x,z,t))^{-1}$$

and from Proposition 4.2 we have $H(x,t) = H_T(x, e^x - 1, t)$. Working out the arithmetic gives the following result.

#P \ Height	0	1	2	3	4	5	6
0	1						
1		1					
2		1	2				
3		1	6	6			
4		1	50	36	24		
5		1	510	510	240	120	
6		1	7682	7380	4800	1800	720

Table 1: The number of strongly graded $(3+1)$-avoiding posets of six or fewer vertices, by height.

Proposition 7.1. *Let $H(x,t)$ be the generating function counting strongly graded $(3+1)$-avoiding posets by number of vertices and height (as in Equation (2)). We have*

$$H(x,t) = \frac{e^x(e^x + te^{2x} + t^2(e^x-1)^2) + t((1-3e^x+e^{2x}) + t(e^x-1)^2(e^x-2))\Psi(x)}{e^x(e^x+te^x+t^2) + t((1-3e^x+e^{2x}) + t(e^x-2))\Psi(x)}.$$

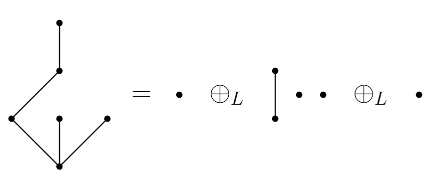
The resulting coefficients are shown in Table 1.

8 Weakly Graded Posets

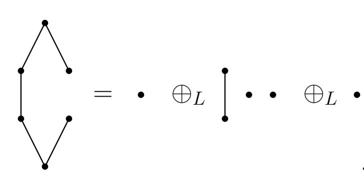
In this section, we expand our study to weakly graded posets. We seek to apply the same methods that worked in the strongly graded case. The results of Section 3, the definition of a trimmed poset, and the results of Section 4.1 carry over immediately to weakly graded posets. We now seek to extend the rest of our work to this context.[4]

We begin by proving Proposition 8.1, which shows that weakly graded $(3+1)$-avoiding posets mostly "look just like" strongly graded posets.

[4] We forewent the opportunity to write the entirety of Section 4 in terms of weakly graded posets. The reason for this choice is as follows: without the condition that posets be $(3+1)$-avoiding, the details of the steps of decomposition become trickier. In particular, in this case it is no longer true that decompositions like $P = P_1 \oplus_L P_2 \oplus_L P_3$ are uniquely reversible when we view the P_i as posets. For example, we have as posets that



and



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i.e., the same two posets can be layered in more than one way. By waiting until this later stage, we manage to avoid this problem by speaking in terms of quarks. For example, in the example above, when we work in terms of quarks we have that each of the isolated vertices in the middle layer knows whether it belongs to the top vertex level or bottom vertex level, removing ambiguity.

Proposition 8.1. *In a weakly graded $(3+1)$-avoiding poset of height k such that $k \geq 3$, all maximal elements are contained on levels $k-1$ and $k-2$ and all minimal elements must be on levels 0 or 1.*

Proof. Let P be a weakly graded $(3+1)$-avoiding poset of height at least 3. A maximal vertex in P is precisely the same as a vertex not comparable to vertices of any higher rank. However, by Theorem 3.1, every vertex of P is comparable to vertices of rank two larger. Putting these two facts together, we immediately conclude that P has no vertices of rank two larger than any of its maximal vertices; this is the desired result. The case of minimal vertices is identical. $\square$

With this result in hand, we can immediately extend the remaining results of Section 4 to weakly graded posets.

Corollary 8.2. *Every trimmed $(3+1)$-avoiding poset P can be decomposed uniquely as a layering of L-indecomposable posets and each of the resulting L-indecomposable posets can be decomposed uniquely by sticking and gluing, where these operations are defined as before, subject to the following changes:*

- *when P is written as a layering of L-indecomposable posets, the topmost and bottommost L-indecomposable posets are weakly graded, not necessarily strongly graded;*
- *when P is written as a layering, sticking and gluing of quarks, the topmost quark may have isolated vertices on its lower vertex level and similarly the bottommost quark may have isolated vertices on its upper vertex level. (These isolated vertices are exactly the maximal vertices not of maximum rank and the minimal vertices not of minimum rank, respectively.)*

This result allows us to directly apply the methods of Section 6.

Observe that any $(3+1)$-avoiding poset with a chain of length 3 or longer must be connected, while posets of height 2 can be somewhat more "wild." This suggests that we should consider separately posets of height 2 or less and posets of height 3 or more. We do this in the following sections.

8.1 Posets of height at most 2

All posets of height at most 2 are weakly graded and avoid $(3+1)$. Of these, there is exactly one with height 0 (the empty poset), and for each $n \geq 1$ there is exactly one poset on n vertices of height 1 (the antichain on n vertices). The number of posets of height 2 on n vertices is precisely $\sum_{m=1}^{n-1} \binom{n}{m} |A^{\otimes}(n-m, m)|$: we choose m vertices to be at rank 1, and none of these can be isolated. It follows from these three cases and from Appendix B that the generating function for weakly graded $(3+1)$-avoiding posets of height at most 2 is

$$1 + t(e^x - 1) + t^2(e^{-x}\Psi(x) - e^x).$$

8.2 Posets of height at least 3

It follows from Corollary 8.2 that we may view a trimmed $(3+1)$-avoiding poset of height at least 3 as a gluing, sticking and layering of quarks subject to the same rules as in Section 6 except that we allow there to be no all-seeing vertex in the bottom vertex level of the poset even when there is an isolated vertex in the top half of the bottommost quark in the construction, and likewise with "top" and "bottom" interchanged.

In other words, we have a bottommost layer that can be encoded as a word with certain compatibility restrictions; the restrictions (and the rules for gluing) are exactly the same as before, but now we want to include words that begin $SB_{\bigcirc}^{\bigcirc}$ or $SB_{\otimes}^{\bigcirc}$. Similarly, we have a top layer encoded as a word with the same restrictions, but now we want to include words that end with $B_{\bigcirc}^{\bigcirc}S$ or $B_{\bigcirc}^{\otimes}S$. Equivalently, we redefine what it means to be a legal word by removing conditions 1 and 2 in Definition 6.1. This corresponds to a straightforward change in the generating function magic of Section 6: the matrices M_W and M that appear in Proposition 6.4 and Corollary 6.5 don't need to change at all, though the vectors by which we multiply on the left and right need to be adjusted.

We now give a detailed plan of action. We handle separately those posets that can and cannot be layered. This gives us two cases:

- posets of height $k+1$, where $k+1>2$, that consist of a single layer with at least one minimal vertex of rank 1 and at least one maximal vertex of rank $k-1$, and
- posets of height $k+1$, where $k+1>2$, that do not fall into the previous class; these posets have an optional top layer with maximal vertices of rank $k-1$, then have a (possibly empty) collection of strongly graded L-indecomposable layers, then have an optional bottom layer with minimal vertices of rank 1.

We will compute the generating functions in these cases following the transfer-matrix approach used previously. Note one important subtlety: in both cases, the transfer-matrix method generates some posets of height 2 or less which we view as spurious. Thus, we use the refined version of the generating functions computed in Section 7 and make sure to eliminate the height-0, 1 and 2 terms in the first two cases. The reason for this approach is closely related to the discussion in Footnote 4: since the transfer-matrix method fundamentally works on quarks, posets with isolated vertices are counted multiple times (once for every possible assignment of the isolated vertices to rank 0 or rank 1).

8.2.1 L-indecomposable posets that cannot be layered

L-indecomposable $(3+1)$-avoiding unlayerable posets of height $k+1$ are exactly those with maximal vertices of rank $k-1$ and minimal vertices of rank 1. Following the line of argument that culminated in Corollary 6.5, we see that the generating function for these posets is precisely

$$t^2 \cdot \begin{bmatrix} F_{\bigcirc}^{\bigcirc} & 0 & F_{\bigcirc}^{\otimes} & 0 \end{bmatrix} \cdot (\mathbb{I}-tM)^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

where

$$M = z \cdot \begin{bmatrix} F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \\ F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \\ F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \\ F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \end{bmatrix} + \begin{bmatrix} 0 & F_{\otimes}^{\bigcirc} & 0 & F_{\otimes}^{\otimes} \\ 0 & F_{\otimes}^{\bigcirc} & 0 & F_{\otimes}^{\otimes} \\ F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \\ F_{\bigcirc}^{\bigcirc} & F_{\otimes}^{\bigcirc} & F_{\bigcirc}^{\otimes} & F_{\otimes}^{\otimes} \end{bmatrix}$$

as before.

8.2.2 All other posets

Trimmed $(3+1)$-avoiding posets not counted in the previous cases have the following structure:

- they may or may not have a top layer (i.e., L-indecomposable poset) with maximal vertices at the level below maximum rank;
- they have some number (possibly 0) of "middle layers" that are strongly graded L-indecomposable posets; and
- they may or may not have a bottom layer with minimal vertices of rank 1.

The generating function for L-indecomposable (3+1)-avoiding posets is the function $H(x, z, t)$ defined in Section 7. We define $\mathrm{top}(x, z, t)$ to be the generating function for L-indecomposable $(3+1)$-avoiding posets with all minimal vertices of rank 0 and with some maximal vertices of non-maximum rank, and analogously we define the generating function $\mathrm{bot}(x, z, t)$. Then the generating function for posets in this class coincides with

$$(1 + \mathrm{top}(x, z, t))(1 - H(x, z, t))^{-1}(1 + \mathrm{bot}(x, z, t)) \tag{3}$$

for all powers of t greater than or equal to 3. Moreover, we have

$$\mathrm{top}(x, z, t) = t^2 \cdot \begin{bmatrix} F_{\bigcirc}^{\bigcirc} & 0 & F_{\bigcirc}^{\otimes} & 0 \end{bmatrix} \cdot (\mathbb{I} - tM)^{-1} \cdot \begin{bmatrix} z \\ z \\ 1+z \\ 1+z \end{bmatrix}$$

and

$$\mathrm{bot}(x, z, t) = t^2 \cdot \begin{bmatrix} zF_{\bigcirc}^{\bigcirc} & (1+z)F_{\otimes}^{\bigcirc} & zF_{\bigcirc}^{\otimes} & (1+z)F_{\otimes}^{\otimes} \end{bmatrix} \cdot (\mathbb{I} - tM)^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

8.3 All weakly graded $(3+1)$-avoiding posets

Finally, we combine the work in the preceding subsections to enumerate weakly graded $(3+1)$-avoiding posets.

Height

#P	0	1	2	3	4	5	6
0	1						
1		1					
2		1	2				
3		1	12	6			
4		1	86	84	24		
5		1	840	1110	480	120	
6		1	11642	16620	9120	3240	720

Table 2: The number of weakly graded $(3+1)$-avoiding posets of six or fewer vertices, by height.

Theorem 8.3. *The generating function for weakly graded* $(3+1)$*-avoiding posets counted by number of vertices is*

$$1 + e^{-x}\Psi(x) + \frac{1 + e^x\Psi(x)}{e^x(2e^x+1) + (e^{2x} - 2e^x - 1)\Psi(x)}.$$

The generating function for weakly graded $(3+1)$*-avoiding posets counted by number of vertices and height is*

$$1 + (e^x - 1)t + (e^{-x}\Psi(x) - e^x)t^2 + \\ t^3\frac{e^{3x} + e^{3x}t - e^x(2e^x + (1 + 2e^x - e^{2x})t)\Psi(x) - ((1 - 3e^x + e^{2x}) + (e^x - 2)t)\Psi(x)^2}{e^x(e^x + e^xt + t^2) + ((1 - 3e^x + e^{2x})t + (e^x - 2)t^2)\Psi(x)}.$$

Proof. The proof is a straightforward (albeit messy) computation: we add the generating functions from Section 8.2.1 to the expression from Equation (3), kill the t^0, t^1 and t^2 terms, and add the result to the generating function from Section 8.1. Mathematica code that carries out these computations may be found in Appendix C. □

The resulting coefficients are shown in Table 2. Disregarding height, for $\#P = 0, 1, \ldots,$ the number of posets of the given size is 1, 1, 3, 19, 195, 2551, 41343, ...

9 Asymptotics

In this section, we compute asymptotics for the number of graded $(3+1)$-avoiding posets. First, we give asymptotics for the coefficients of the series $\Psi(x)$; then we give the asymptotics for our posets in terms of the asymptotics of Ψ. Define $\psi_n = \sum_{i=0}^{n} \frac{2^{i(n-i)}}{i!(n-i)!}$ so that $\Psi(x) = 1 + \sum_{n\geq 1}\psi_n x^n$. In the next result, we give asymptotics for the coefficients ψ_n.

Proposition 9.1. *There exist constants* C_1 *and* C_2 *such that*

$$\psi_{2k} \sim C_1 \cdot \frac{2^{k^2}}{(k!)^2} \qquad \textit{and} \qquad \psi_{2k+1} \sim C_2 \cdot \frac{2^{k(k+1)}}{k!(k+1)!}.$$

Proof. For $n = 2k$ even, we can write

$$
\begin{aligned}
\psi_{2k} &= \sum_{i=0}^{2k} \frac{2^{i(2k-i)}}{i!(2k-i)!} \\
&= \frac{2^{k^2}}{(k!)^2} \left(1 + 2 \sum_{i=0}^{k-1} 2^{i(2k-i)-k^2} \frac{(k!)^2}{i!(2k-i!)} \right) \\
&= \frac{2^{k^2}}{(k!)^2} \left(1 + 2 \sum_{i=0}^{k-1} 2^{-(i-k)^2} \frac{(k!)^2}{i!(2k-i!)} \right) \\
&= \frac{2^{k^2}}{(k!)^2} \left(1 + 2 \sum_{i=1}^{k} 2^{-i^2} \frac{(k!)^2}{(k-i)!(k+i)!} \right) \\
&= \frac{2^{k^2}}{(k!)^2} \left(1 + \frac{2}{2^1} \cdot \frac{k}{k+1} + \frac{2}{2^4} \cdot \frac{k(k-1)}{(k+1)(k+2)} + \frac{2}{2^9} \cdot \frac{k(k-1)(k-2)}{(k+1)(k+2)(k+3)} + \cdots \right)
\end{aligned}
$$

This is asymptotically equivalent to $C_1 \frac{2^{k^2}}{(k!)^2}$ with $C_1 = 1 + 2(2^{-1} + 2^{-4} + 2^{-9} + \cdots)$, as claimed. (Incidentally, $C_1 \approx 2.12893682721\ldots$ is the value of a Jacobi theta function.)

For $n = 2k + 1$ odd, we can make a similar calculation:

$$
\begin{aligned}
\psi_{2k+1} &= \sum_{i=0}^{2k+1} \frac{2^{i(2k+1-i)}}{i!(2k+1-i)!} \\
&= \frac{2^{k(k+1)}}{k!(k+1)!} \left(2 + 2 \sum_{i=0}^{k-1} 2^{i(2k+1-i)-k(k+1)} \frac{k!(k+1)!}{i!(2k+1-i!)} \right) \\
&= \frac{2^{k(k+1)}}{k!(k+1)!} \left(2 + 2 \sum_{i=0}^{k-1} 2^{-(k-i)(k-i+1)} \frac{k!(k+1)!}{i!(2k+1-i!)} \right) \\
&= \frac{2^{k(k+1)}}{k!(k+1)!} \left(2 + 2 \sum_{i=1}^{k} 2^{-i(i+1)} \frac{k!(k+1)!}{(k-i)!(k+1+i)!} \right) \\
&= \frac{2^{k(k+1)}}{k!(k+1)!} \left(2 + \frac{2}{2^2} \cdot \frac{k}{k+2} + \frac{2}{2^6} \cdot \frac{k(k-1)}{(k+2)(k+3)} + \frac{2}{2^{12}} \cdot \frac{k(k-1)(k-2)}{(k+2)(k+3)(k+4)} + \cdots \right)
\end{aligned}
$$

This is asymptotically equivalent to $C_2 \cdot \frac{2^{k(k+1)}}{k!(k+1)!}$ with $C_2 = 2(1 + 2^{-2} + 2^{-6} + 2^{-12} + \cdots) \approx 2.5317401904617\ldots$. ∎

Let g_n be the number of strongly graded $(3+1)$-avoiding posets on n vertices and let w_n be the number of weakly graded $(3+1)$-avoiding posets on n vertices.

Theorem 9.2. *We have $g_n \sim n! \cdot \psi_n$ and $w_n \sim 6 \cdot n! \cdot \psi_n$.*

The proof relies on the following special case of a theorem of Bender [2], which may also be found in [8, Theorem 7.3]:

Theorem 9.3 ([2, Theorem 1])**.** *Suppose* $A(x) = \sum_{n\geq 1} a_n x^n$, *that* $F(x,y)$ *is a formal power series in* x *and* y, *and that* $B(x) = \sum_{n\geq 0} b_n x^n = F(x, A(x))$. *Let* $C = \frac{\partial}{\partial y} F\Big|_{(0,0)}$. *Suppose further that*

1. $F(x,y)$ *is analytic in a neighborhood of* $(0,0)$,
2. $\lim_{n\to\infty} \dfrac{a_{n-1}}{a_n} = 0$, *and*
3. $\displaystyle\sum_{k=1}^{n-1} |a_k a_{n-k}| = O(a_{n-1})$.

Then

$$b_n = C \cdot a_n + O(a_{n-1})$$

and in particular $b_n \sim C a_n$.

We now use this result to prove Theorem 9.2.

Proof. Define

$$F_1(x,y) = 1 + \frac{e^{2x}(2e^x - 3) + e^x(e^x - 2)^2(y+1)}{e^x(2e^x + 1) + (e^{2x} - 2e^x - 1)(y+1)}$$

and

$$F_2(x,y) = 1 + e^{-x}(y+1) + \frac{1 + e^x(y+1)}{e^x(2e^x + 1) + (e^{2x} - 2e^x - 1)(y+1)}$$

so that

$$F_1(x, \Psi(x) - 1) = G(x)$$

is the exponential generating function for strongly graded $(3+1)$-avoiding posets (compare Theorem 6.7) and

$$F_2(x, \Psi(x) - 1) = W(x)$$

is the exponential generating function for weakly graded $(3+1)$-avoiding posets (compare Theorem 8.3). (Here the off-by-one is so that the composition is formally valid.) We seek to apply Theorem 9.3 to compute asymptotics for the coefficients of G and W. To apply the theorem, we have three conditions to check. The first condition is that F_1 and F_2 are analytic in a neighborhood of $(0,0)$, which is clear by inspection. The second condition follows immediately from Proposition 9.1. The third condition is slightly trickier: we must show that

$$\sum_{k=1}^{n-1} \psi_k \psi_{n-k} = O(\psi_{n-1}).$$

We do that now.

The proof of Proposition 9.1 not only gives asymptotics for ψ_n but also shows that

$$\psi_n \leq C \frac{2^{n^2/4}}{\lfloor n/2 \rfloor! \cdot \lceil n/2 \rceil!}$$

for all n. In addition, by taking only one term of the sum we have

$$\psi_{n-1} \geq \frac{2^{\lfloor (n-1)/2 \rfloor \cdot \lceil (n-1)/2 \rceil}}{\lfloor (n-1)/2 \rfloor! \cdot \lceil (n-1)/2 \rceil!}$$

for all n. Thus

$$\sum_{k=1}^{n-1} \frac{\psi_k \psi_{n-k}}{\psi_{n-1}} \leq C' \sum_{k=1}^{n-1} \frac{\lfloor (n-1)/2 \rfloor! \cdot \lceil (n-1)/2 \rceil! \cdot 2^{-(k-1)(n-k-1)/2}}{\lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! \cdot \lfloor (n-k)/2 \rfloor! \cdot \lceil (n-k)/2 \rceil!}$$

for some constant C', and we wish to show that this last summation is bounded by an absolute constant independent of n. The $k = 1$ and $k = n - 1$ terms of the sum are constant in n while the $k = 2$ and $k = n - 2$ terms contribute a combined

$$O\left(\lceil (n-1)/2 \rceil 2^{-n/2}\right) = o(1)$$

to the sum. Each of the remaining terms is bounded by

$$\frac{\lfloor (n-1)/2 \rfloor! \cdot \lceil (n-1)/2 \rceil! \cdot 2^{-n+4}}{(n/4)!^4} = O(n^{-2})$$

and so the total contribution of these terms is also $o(1)$. Thus

$$\sum_{k=1}^{n-1} \frac{\psi_k \psi_{n-k}}{\psi_{n-1}}$$

is bounded above by a constant independent of n, as desired.

We've shown that the conditions of Theorem 9.3 hold and we now apply it directly. By direct computation,

$$\frac{\partial}{\partial y} F_1(x, y) = \frac{e^{2x}}{((3e^{2x} - e^x - 1) + (e^{2x} - 2e^x - 1)y)^2}.$$

and

$$\frac{\partial}{\partial y} F_2 = e^{-x} + \frac{2e^{3x} + 2e^x + 1}{((3e^{2x} - e^x - 1) + (e^{2x} - 2e^x - 1)y)^2}.$$

Thus $\frac{\partial}{\partial y} F_1(x, y)\Big|_{(0,0)} = 1$ and $\frac{\partial}{\partial y} F_2(x, y)\Big|_{(0,0)} = 6$ and so

$$\frac{g_n}{n!} = \psi_n + O(\psi_{n-1}) \sim \psi_n \qquad \text{and} \qquad \frac{w_n}{n!} = 6\psi_n + O(\psi_{n-1}) \sim 6\psi_n,$$

as desired. □

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A Some useful enumerative lemmas

In this appendix, we mention without proof certain enumerative results from the theory of species as they apply in our setting; this simplifies what would otherwise be arduous checking of details. A brief, excellent exposition of these results in the general setting may be found in [6, Section 4]; for a more detailed treatment, see [5, Chapter 3].

Suppose that $\mathcal{A}_0, \mathcal{A}_1, \ldots$ are classes of combinatorial objects with two types of labeled vertices, $\mathbf{s}$-vertices and $\mathbf{t}$-vertices. Each such class has a naturally associated generating function: for a given $X \in \mathcal{A}_i$, let $s(X)$ be the number of $\mathbf{s}$-vertices and let $t(X)$ be the number of $\mathbf{t}$-vertices; then $\mathcal{A}_i$ is enumerated by the exponential generating function

$$A_i(x,z) = \sum_{X \in \mathcal{A}_i} \frac{x^{s(X)} z^{t(X)}}{s(X)!t(X)!}. \tag{4}$$

We now consider several possible combinatorial relationships between the objects of the $\mathcal{A}_i$ and state the resulting relationships between the A_i.

First, suppose that $\mathcal{A}_0$ is a class of combinatorial objects that arise in the following way: for some fixed k, each element of $\mathcal{A}_0$ is the result of choosing elements $X_1 \in \mathcal{A}_1$, $X_2 \in \mathcal{A}_2$, $\ldots$, and $X_k \in \mathcal{A}_k$, attaching them together in some canonical way, and then relabeling the vertices of the resulting object in a way that is consistent with the labelings of the subobjects. Then we write $\mathcal{A}_0 = \mathcal{A}_1 \star \mathcal{A}_2 \star \cdots \star \mathcal{A}_k$. We have the following result.

Lemma A.1 ($\star$-product)**.** *Let the operation $\star$ be defined as in the preceding paragraph. If $\mathcal{A}_0 = \mathcal{A}_1 \star \mathcal{A}_2 \star \cdots \star \mathcal{A}_k$ then*

$$A_0(x,z) = \prod_{i=1}^{k} A_i(x,z).$$

For example, suppose that $\mathcal{A}_1$ is the set of nonempty antichains on two types of vertices and that $\mathcal{A}_0$ is the set of graded posets of height k on the same two types of vertices such that every vertex is comparable to all vertices of higher or lower ranks. Then $\mathcal{A}_0 = \mathcal{A}_1 \star \mathcal{A}_1 \star \cdots \star \mathcal{A}_1$. We have $A_1(x,z) = e^{x+z} - 1$ and so $A_0(x,z) = (e^{x+z} - 1)^k$.

Second, suppose that $\mathcal{A}_0$ is a class of combinatorial objects that arise in the following way: each element of $\mathcal{A}_0$ is the result of choosing an element $X_1 \in \mathcal{A}_1$, replacing each of its $\mathbf{t}$-vertices with a nonempty element of $\mathcal{A}_2$ in some canonical way, and then relabeling the vertices of the resulting object in a way that is consistent with the labelings on the subobjects. Then we write $\mathcal{A}_0 = \mathcal{A}_1 \circ \mathcal{A}_2$. We have the following result.

Lemma A.2 (Composition)**.** *Let the operation $\circ$ be defined as in the preceding paragraph. If $\mathcal{A}_0 = \mathcal{A}_1 \circ \mathcal{A}_2$ then*

$$A_0(x,z) = A_1(x, A_2(x,z)).$$

For example, suppose that $\mathcal{A}_1$ is the set of chains on $\mathbf{s}$- and $\mathbf{t}$-vertices and $\mathcal{A}_2$ is the set of antichains of size 2 or more, all of whose vertices are $\mathbf{s}$-vertices. Then $\mathcal{A}_0 = \mathcal{A}_1 \circ \mathcal{A}_2$ is the set of graded posets on $\mathbf{s}$-vertices such that every vertex is comparable to all vertices of higher and lower ranks. We have $A_1(x,z) = \sum_{s,t\geq 0}(s+t)!\frac{x^s z^t}{s!t!} = \frac{1}{1-x-z}$ and $A_2(x,z) = \sum_{s\geq 2}\frac{x^s}{s!} = e^x - x - 1$, so by Lemma A.2, $A_0(x,z) = \frac{1}{2-e^x}$. We can get the same result by taking $\mathcal{A}_3$ to

be the set of chains of **t**-vertices and $\mathcal{A}_4$ to be the set of nonempty antichains of **s**-vertices. Then we have $\mathcal{A}_0 = \mathcal{A}_3 \circ \mathcal{A}_4$, $A_3(x,z) = \frac{1}{1-z}$, $A_4(x,z) = e^x - 1$ and so $A_0(x,z) = \frac{1}{2-e^x}$ as before.

Finally, we remark that in the body of this paper we treat the generating function $G_T(x,z)$ (and similarly $I(x,z)$, etc.) as exponential in x and ordinary in z, while in the preceding lemmas the generating functions used are exponential in both variables. The applicability of our lemmas in this case is justified by the following discussion and lemma. Suppose that $\mathcal{A}_0$ is a class of combinatorial objects with two types of labeled vertices, **s**-vertices and **t**-vertices, and suppose $\mathcal{A}_1$ is the associated class of combinatorial objects in which we identify elements of $\mathcal{A}_0$ that differ only by relabelings of the **t**-vertices. Then $\mathcal{A}_1$ has a natural generating function

$$A_1(x,z) = \sum_{X \in \mathcal{A}_1} \frac{x^{s(X)}}{s(X)!} z^{t(X)}$$

that is exponential in x but ordinary in z, while $\mathcal{A}_0$ still has the associated generating function defined by Equation (4). We have the following result.

Lemma A.3. *Let $\mathcal{A}_0$, $\mathcal{A}_1$, A_0 and A_1 be defined as in the preceding paragraph. Suppose furthermore that for every $X \in \mathcal{A}_0$, the **t**-vertices of X are distinct, i.e., every automorphism of X fixes all **t**-vertices. Then*

$$A_0(x,z) = A_1(x,z).$$

For example, if $\mathcal{A}_0$ is the set of chains on the two types of labeled vertices then

$$A_0(x,z) = \sum_{s,t \geq 0} (s+t)! \frac{x^s z^t}{s!t!}.$$

In this case $\mathcal{A}_1$ is the set of chains on a collection of some labeled and some unlabeled vertices, and so

$$A_1(x,z) = \sum_{s,t \geq 0} \binom{s+t}{t} s! \frac{x^s}{s!} z^t.$$

It's clear that $A_0(x,z) = A_1(x,z)$, as claimed.

B Computing generating functions for quarks

In this appendix, we enumerate and compute generating functions counting those sets of the form A_μ^ν (introduced in Section 5) that are of use to us. For bookkeeping purposes, we make these generating functions bivariate in variables x and y, with each bipartite graph in $A_\mu^\nu(m,n)$ (i.e., each graph on the vertex set $[m] \uplus [n]$ with appropriate restrictions) giving a contribution of $\frac{x^m y^n}{m!n!}$.

It is very convenient to introduce the generating function

$$\Psi(x,y) = \sum_{m,n \geq 0} \frac{2^{mn} x^m y^n}{m!n!},$$

as most of our generating functions are most easily expressed in terms of $\Psi(x,y)$.

1. $|A(m,n)| = 2^{mn}$: we have no restrictions, so all of the mn edges may choose independently to be present or absent. Equivalently, we have $\sum_{m,n\geq 1}|A(m,n)|\frac{x^m y^n}{m!n!} = \Psi(x,y) - e^x - e^y + 1$. (The extra terms at the end simply account for the fact that we sum here only over positive values of m, n.)

2. $|A_{\boxtimes}(m,n)| = |A_{\otimes}(m,n)| = (2^n-1)^m$: we need every vertex on the m-side to be not all-seeing (respectively, isolated) and there are no other restrictions. It's not hard to compute the generating function

$$\sum_{m,n\geq 1}|A_{\otimes}(m,n)|\frac{x^m y^n}{m!n!} = e^{-x}\Psi(x,y) - e^y.$$

   It follows by symmetry that the generating function for $A^{\boxtimes}$ and $A^{\otimes}$ is $e^{-y}\Psi(x,y) - e^x$.

3. $|A_{\boxtimes\otimes}(m,n)| = (2^n-2)^m$: each vertex on the m-side can be connected to any subset on the n-side except the empty set or everything. The associated generating function is $e^{-2x}\Psi(x,y) - e^{-x} - e^y + 1$.

4. $|A^{\boxtimes}_{\boxtimes}(m,n)| = |B(m,n)|$: This is the first tricky example. First, we show that

$$|B(m,n)| = \sum_{i=0}^{m}(-1)^i\binom{m}{i}(2^{m-i}-1)^n. \tag{5}$$

   The proof is by inclusion-exclusion on the all-seeing vertices in $[n]$. For a subset $S \subseteq [n]$, the number of graphs in which the vertices of S are all-seeing and no vertices of $[m]$ are all-seeing is $(2^{n-|S|}-1)^m$: each vertex in $[m]$ may choose the union of S with any proper subset of $[n] \setminus S$ to be its neighbors, and these choices may be made independently. Applying inclusion-exclusion immediately gives the result. (As an aside, this means that the summation expression on the right-hand side of Equation (5) is symmetric in m and n, a fact not immediately obvious from its formula.)

   Now, we can do the calculation

$$\begin{aligned}
1 + \sum_{m,n\geq 1}|B(m,n)|\frac{x^m y^n}{m!n!} &= \sum_{m,n\geq 0}|B(m,n)|\frac{x^m y^n}{m!n!} \\
&= \sum_{m,n\geq 0}\sum_{i=0}^{m}(-1)^i\binom{m}{i}(2^{m-i}-1)^n\frac{x^m y^n}{m!n!} \\
&= \sum_{m,n\geq 0}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{m-i}(2^{i}-1)^n\frac{x^m y^n}{m!n!} \\
&= \sum_{m,n\geq 0}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{m+n-i-j}\binom{m}{i}\binom{n}{j}2^{ij}\frac{x^m y^n}{m!n!} \\
&= \sum_{m,n\geq 0}\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{m+n-i-j}\frac{x^{m-i}y^{n-j}}{(m-i)!(n-j)!}\cdot\frac{2^{ij}x^i y^j}{i!j!} \\
&= e^{-x-y}\Psi(x,y).
\end{aligned}$$

5. $|A^{\boxtimes}_{\boxtimes\bigcirc}(m,n)| = |B_{\bigcirc}(m,n)|$: From the definitions of the sets A^{ν}_{μ} and the preceding computations we have

$$\begin{aligned}
|A^{\boxtimes}_{\boxtimes\bigcirc}| &= |A_{\boxtimes\bigcirc}| - |A^{\square}_{\boxtimes\bigcirc}| \\
&= |A_{\boxtimes\bigcirc}| \\
&= |A_{\boxtimes}| - |A_{\boxtimes\otimes}| \\
&= (2^n-1)^m - (2^n-2)^m.
\end{aligned}$$

The associated generating function is $(1-e^{-x})(e^{-x}\Psi(x,y)-1)$.

6. $|A^{\boxtimes\bigcirc}_{\boxtimes\bigcirc}(m,n)| = |B^{\bigcirc}_{\bigcirc}(m,n)|$. We have

$$\begin{aligned}
|A^{\boxtimes\bigcirc}_{\boxtimes\bigcirc}| &= |A^{\boxtimes\bigcirc}_{\bigcirc}| - |A^{\boxtimes\bigcirc}_{\square\bigcirc}| \\
&= |A^{\boxtimes\bigcirc}_{\bigcirc}| \\
&= |A^{\bigcirc}_{\bigcirc}| - |A^{\square\bigcirc}_{\bigcirc}| \\
&= |A^{\bigcirc}_{\bigcirc}| \\
&= |A^{\bigcirc}| - |A^{\bigcirc}_{\otimes}| \\
&= (|A| - |A^{\otimes}|) - (|A_{\otimes}| - |A^{\otimes}_{\otimes}|) \\
&= |A| - |A^{\otimes}| - |A_{\otimes}| + |B|.
\end{aligned}$$

Combining our previous results, we have that the associated generating function is $(1-e^{-x})(1-e^{-y})\Psi(x,y)$.

Finally, we may use the work above to compute the generating functions we desire. We have

$$\begin{aligned}
F^{\bigcirc}_{\bigcirc}(x) &= \sum_{m,n\geq 1} |B^{\bigcirc}_{\bigcirc}(m,n)| \frac{x^{m+n}}{m!n!} \\
&= (1-e^{-x})^2\Psi(x,x), \\
F^{\otimes}_{\bigcirc}(x) &= \sum_{m,n\geq 1} |B^{\otimes}_{\bigcirc}(m,n)| \frac{x^{m+n}}{m!n!} \\
&= \sum_{m,n\geq 1} \big(|B_{\bigcirc}(m,n)| - |B^{\bigcirc}_{\bigcirc}(m,n)|\big) \frac{x^{m+n}}{m!n!} \\
&= (1-e^{-x})(e^{-x}\Psi(x,x)-1) - (1-e^{-x})^2\Psi(x,x) \\
&= (1-e^{-x})((2e^{-x}-1)\Psi(x,x)-1),
\end{aligned}$$

and similarly

$$F^{\bigcirc}_{\otimes}(x) = (1-e^{-x})((2e^{-x}-1)\Psi(x,x)-1)$$

and

$$F^{\otimes}_{\otimes}(x) = (2e^{-x}-1)((2e^{-x}-1)\Psi(x,x)-1).$$

C Arithmetic

The calculations used to derive Theorem 6.7 and Proposition 7.1 correspond to the following Mathematica code:

```
(* quarks *)
F = Exp[-2 x] Psi[x] - 1;
Fsubo = (1 - Exp[-x]) (Exp[-x] Psi[x] - 1);
Fsupo = (1 - Exp[-x]) (Exp[-x] Psi[x] - 1);
Boo = (1 - Exp[-x])^2 Psi[x];
Fox = Fsupo - Foo;
Fxo = Fsubo - Foo;
Fxx = F - Fsubo - Fsupo + Foo;

(* transfer matrix *)
Mat = {{z, 1 + z, z, 1 + z}, {z, 1 + z, z, 1 + z},
       {1 + z, 1 + z, 1 + z, 1 + z}, {1 + z, 1 + z, 1 + z, 1 + z}}.
      {{Foo, 0, 0, 0}, {0, Fox, 0, 0}, {0, 0, Fxo, 0}, {0, 0, 0, Fxx}};

(* L-indecomposables *)
Ind = t^2 ({{z*Foo, (1 + z)*Fox, z*Fxo, (1 + z)*Fxx}}.
       MatrixPower[IdentityMatrix[4] - t*Mat, -1].
       {{z}, {z}, {1 + z}, {1 + z}})[[1, 1]];

(* trimmed to normal *)
z = Exp[x] - 1;

(* layering L-indecomposables *)
Simplify[(1 - t*z - Ind)^(-1) /. t->1]

(* the first few terms of the sequence *)
k = 8;
Series[%/. Psi[x]->Sum[Sum[2^(m n) x^(m + n)/(m! n!), {n, 0, k - m}],
       {m, 0, k}], {x, 0, k}];
CoefficientList[%,x]*Range[0, k]!

(* taking height into consideration *)
k=8;
TableForm[
 Map[CoefficientList[#,t]&,
  CoefficientList[Series[
   (1 - t*z - Ind)^(-1) /.
    Psi[x]->Sum[Sum[2^(m n) x^(m + n)/(m! n!), {n, 0, k - m}], {m, 0, k}],
    {x,0,k}],x]*Range[0,k]!]]
```

The additional computations in the proof of Theorem 8.3 correspond to the following Mathematica code:

```
(* weakly graded *)
top = t^2 ({{Foo, 0, Fxo, 0}}.
          MatrixPower[IdentityMatrix[4] - t*Mat, -1].
          {{z}, {z}, {1 + z}, {1 + z}})[[1,1]];
bottom = t^2 ({{z*Foo, (1+z)Fox, z*Fxo, (1+z)Fxx}}.
             MatrixPower[IdentityMatrix[4] - t*Mat, -1].
             {{1}, {1}, {0}, {0}})[[1,1]];
unlayered = t^2 ({{Foo, 0, Fxo, 0}}.
                MatrixPower[IdentityMatrix[4] - t*Mat, -1].
                {{1}, {1}, {0}, {0}})[[1,1]];
short = 1 + t(Exp[x] - 1) + t^2(Exp[-x]Psi[x] - Exp[x]);
Simplify[((1 + top)(1 - t*z - Ind)^(-1)(1 + bottom) + unlayered) -
          Normal[Series[
              ((1 + top)(1 - t*z - Ind)^(-1)(1 + bottom) + unlayered),
              {t, 0, 2}]] + short]
Simplify[%/.t->1]
```

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